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SHORTER COMMUNICATIONS

ON THE THERMAL STABILITY OF A FROZEN CRUST IN FORCED FLOW ON AN INSULATED FINITE WALL†

MOHAMED S. EL-GENK‡

Chemical and Nuclear Engineering Department, University of New Mexico, Albuquerque, NM 87313, U.S.A.

and

AUGUST W. CRONENBERG

Thermal Fuel Behavior Program, EG & G Idaho, Inc., P.O. Box 1625, Idaho Falls, ID 83401, U.S.A.

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NOMENCLATURE

a, wall thickness;
C_p, specific heat;
h, convective coefficient of heat transfer;
k, thermal conductivity;
L, latent heat of fusion;
SN, Stefan number for freezing;
t, time;
t_a, time at which the thermal front in the wall reaches the outside surface, $x = -a$;
T, temperature;
T_0, initial temperature of the wall;
x, distance measured from the wall–crust interface.

Greek symbols

α, thermal diffusivity;
δ, instantaneous frozen crust thickness;
δ_T, instantaneous thickness of the thermal layer in the wall;
Δ, dimensionless thickness of the solidified crust;
ΔaT, dimensionless thickness of the wall;
Δ_T, dimensionless thickness of the wall thermal layer;
θ_a, dimensionless temperature at the wall surface;
θ_I, dimensionless temperature at the wall–crust interface;
θ_s, dimensionless temperature in the frozen crust;
θ_w, dimensionless temperature in the wall;
ε, inverse Stefan number for freezing;
σ, wall–frozen crust thermal ratio;
ρ, density;
τ, dimensionless time;
τ_a, dimensionless time at which the thermal front, δ_T, reaches the wall outer surface, $x = -a$;
τ^*, dimensionless time during the second stage of the analysis ($\tau^* = \tau - \tau_a$);
τ_{life}, total lifetime.

Subscripts

a, wall outside surface;
b, freezing liquid bulk;
f, fusion;
I, at the interface between the wall and the frozen crust;
max, at the maximum thickness of the frozen crust;
s, solidified crust;
w, solid wall;
l, freezing liquid.

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‡ Present address: Thermal Fuels Behavior Program EG & G Idaho, Inc., P.O. Box 1625, Idaho Falls, ID 83401, U.S.A.

1. INTRODUCTION

IN A RECENT paper [1] the growth (freezing) and decay (melting) behavior of the frozen crust that forms in forced flow on a semi-infinite solid wall was studied analytically using the integral heat balance method, as formulated by Goodman [2]; it was found to depend upon the wall–crust thermal ratio, $(\rho_w k_w C_{p_w}/\rho_s k_s C_{p_s})^{1/2}$, and the Stefan number for freezing, $C_{p_s}(T_f - T_0)/L$. Presently available solutions in a finite geometry consider the wall to be isothermal [3–9], where the frozen crust continues to grow in thickness approaching a steady-state value when the convective heat flux from the liquid balances that which can be conducted away through the wall. A solution for the transient growth and decay behavior of the frozen crust in finite geometry has not been developed to date.

In the present work, an approximate solution is presented for the growth and decay behavior of the frozen layer that forms in forced flow on a finite non-melting wall, subject to an adiabatic boundary condition at its opposite surface. The refined integral heat balance (RIHB) method, as introduced in a previous paper [10], is used in the present analysis. Such a method has been shown to provide an accurate prediction of the instantaneous position of the moving change-of-phase front, as applied to one-dimensional melting and freezing problems.

2. PHYSICAL MODEL

As shown in Fig. 1, the problem considered is that of the freezing of a flowing liquid on a cold, non-melting wall of finite thickness, $-a \leq x \leq 0$. The liquid is at a fixed bulk temperature, T_b, above its freezing point, T_f, whereas the wall is initially at some uniform temperature, T_0, below T_f and subject to an adiabatic boundary condition at its outside surface, $x = -a$. The conditions imposed upon the problem are:

1. Axial heat conduction in the frozen crust is neglected.
2. The thermophysical properties of the liquid, the solidified crust and the wall are constant, but different.
3. The convective coefficient of heat transfer at the liquid–solidified crust interface, h, is assumed constant, where the effect of the moving change-of-phase front on disturbing

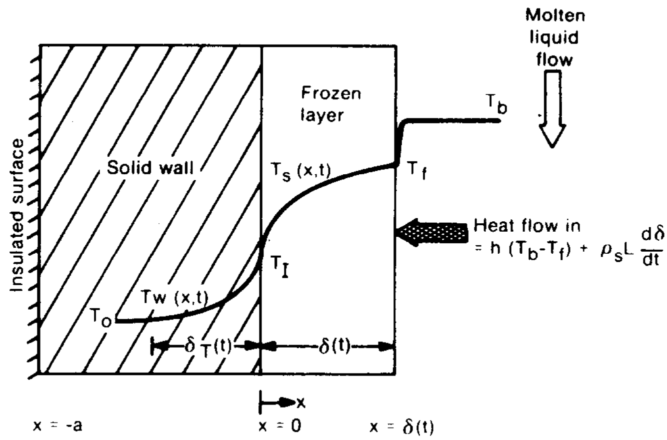


FIG. 1. Schematic diagram for the freezing of a flowing liquid onto a cold wall of finite thickness, $-a \leqslant x \leqslant 0$, the wall being insulated at $x = -a$.

the thermal boundary layer in the liquid is neglected.

The analysis of the present problem is handled in two stages:

First stage

During the time period when the thermal penetration distance is less than the physical dimension of the wall (that is, $\delta_T < a$), the mathematical formulation of the heat transfer problem is the same as that for the freezing of a flowing liquid on a semi-infinite wall [1]. The transient heat conduction equation for the solidified crust and the wall is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \tag{1}$$

while the boundary conditions at the solidification and the thermal penetration fronts are

$$T_s(\delta, t) = T_f,$$

$$k_s \frac{\partial T_s}{\partial x}(\delta, t) = h(T_b - T_f), + \rho_s L \frac{d\delta}{dt}, \tag{2a}$$

and

$$T_w(-\delta_T, t) = T_0, \quad k_w \frac{\partial T_w}{\partial x}(-\delta_T, t) = 0. \tag{2b}$$

The boundary conditions to be satisfied at $x = 0$ are

$$T_s(0,t) = T_w(0,t), \text{ and } k_s \frac{\partial T_s}{\partial x}(0,t) = k_w \frac{\partial T_w}{\partial x}(0,t). \tag{3}$$

The parameters to be calculated during this stage are the transient frozen layer thickness, $\delta(t)$, the thermal penetration distance into the wall, $\delta_T(t)$, and the temperature at the common plane of separation, $T_I(t)$. The initial conditions are that $\delta(t = 0) = 0$ and $\delta_T(t = 0) = 0$.

Second stage

During the second stage of the analysis the thermal layer front, $\delta_T(t)$, reaches the wall surface such that $\delta_T(t = t_a) = a$. The governing equations in both the crust and the wall are the same as in the first stage but the boundary conditions in the wall becomes

$$T_w(-a, t^*) = T_a(t^*) \text{ and } k_w \frac{\partial T_w}{\partial x}(-a, t^*) = 0,$$

$$\text{where } (t^* = t - t_a). \tag{4}$$

The transient parameters to be calculated during this stage are the frozen layer thickness, $\delta(t^*)$, the interface temperature, $T_I(t^*)$, and the wall surface temperature, $T_a(t^*)$. The initial conditions are that $\delta(t^* = 0) = \delta(t_a)$, and $T_a(t^* = 0) = T_0$. Initial condition for the frozen crust thickness, δ, and the value for the interface temperature, T_I, at $t = t_a$ are calculated during the first stage of the analysis.

3. METHOD OF SOLUTION

Since the original concept of the integral method [2, 10], as applied to solidification and melting problems, is well documented in the literature, the method of solution is only briefly outlined here. The basic steps are similar to those described in [1], with some modification to the integration scheme as suggested in [10]. The reader is referred to those two papers [1, 10] as a guide to the solution procedure. The following dimensionless parameters are used herein for simplification of the analysis:

$$\tau \equiv \left[\frac{h^2(T_b - T_f)^2}{k_s^2(T_f - T_0)^2}(\alpha_s t)\right]$$
(dimensionless time);

$$\tau_a \equiv \left[\frac{h^2(T_b - T_f)^2}{k_s^2(T_f - T_0)^2}(\alpha_s t_a)\right]$$
(dimensionless time at which the thermal front δ_T reaches the outside surface, $x = -a$);

$$\Delta \equiv \left[\frac{h(T_b - T_f)}{k_s(T_f - T_0)}\delta(t)\right]$$
(dimensionless frozen crust thickness);

$$\Delta_T \equiv \left[\left(\frac{\alpha_s}{\alpha_w}\right)^{1/2} \frac{h(T_b - T_f)}{k_s(T_f - T_0)}\delta_T\right]$$
(dimensionless wall thermal penetration thickness);

$$\Delta_{aT} \equiv \left[\left(\frac{\alpha_s}{\alpha_w}\right)^{1/2} \frac{h(T_b - T_f)}{k_s(T_f - T_0)}a\right]$$
(dimensionless wall thickness);

$$\theta_a \equiv \left[\frac{T_a - T_0}{T_f - T_0}\right]$$
(dimensionless wall surface temperature);

$$\theta_s \equiv \left[\frac{T_s - T_0}{T_f - T_0}\right]$$
(dimensionless temperature in the frozen layer);

$$\theta_I \equiv \left[\frac{T_I - T_0}{T_f - T_0}\right]$$
(dimensionless wall–crust interface temperature);

$$\theta_w \equiv \left[\frac{T_w - T_0}{T_f - T_0}\right]$$
(dimensionless temperature in the solid wall);

$$\varepsilon \equiv \frac{L}{Cp_s(T_f - T_0)}$$
(inverse Stefan number for freezing);

$$\sigma \equiv \left(\frac{k_w \rho_w Cp_w}{k_s \rho_s Cp_s}\right)^{1/2}$$
(wall–frozen crust thermal ratio). (5)

As usual with the integral heat balance method, the temperature field in the frozen layer during the first stage of the analysis (that is for $t < t_a$) is assumed to be described by a second degree polynominal in time and space of the following form:

$$\theta_s(x,t) = 1 - (1-\theta_I)\left\{B(t)\left(1 - \frac{x}{\delta(t)}\right) + [1 - B(t)]\left(1 - \frac{x}{\delta(t)}\right)^2\right\}, \quad (6)$$

where $B(t)$ is a shape function dependent upon the boundary conditions in the solidified crust. The peculiarity of the present problem is that the temperature at the wall–crust interface is not constant. As a result, the heat-transfer process in the frozen layer must be coupled to that in the solid wall. To accomplish this coupling, the temperature field in the wall is also assumed as a second degree polynominal in time and space of the form:

$$\theta_w(x,t) = \theta_I\left(1 + \frac{x}{a}\right)^2. \quad (7)$$

From equations (1)–(3), (5)–(7), the following coupled system of dimensionless equations is obtained:

$$\varepsilon\frac{d\Delta}{d\tau} = 2\left(\frac{1-\theta_I}{\Delta}\right) - 2\left(\frac{\sigma}{\Delta_T}\right)\theta_I - 1,$$

$$\frac{d\Delta_T}{d\tau} = \left(\frac{6}{\Delta_T}\right) - \left(\frac{\Delta_T}{2\theta_I}\right)\frac{d\theta_I}{d\tau},$$

and

$$\frac{d\theta_I}{d\tau} = a_{10}\frac{d\Delta}{d\tau} + a_{20}\frac{d\Delta_T}{d\tau} - a_{30}, \quad (8)$$

where

$$a_{10} = a_1/a_0, \quad a_{20} = a_2/a_0, \quad a_{30} = a_3/a_0,$$

$$a_0 = \Delta\left[\frac{3}{2}\frac{\Delta}{\Delta_T} + \sigma\left(\frac{\Delta}{\Delta_T}\right)^2\right],$$

$$a_1 = 3\left[(2\varepsilon + 1) - \theta_I\left(1 + \sigma\frac{\Delta}{\Delta_T}\right)\right]\left(\frac{\Delta}{\Delta_T}\right),$$

$$a_2 = \left[\sigma\theta_I\left(\frac{\Delta}{\Delta_T}\right)^3\right], \quad (9)$$

and

$$a_3 = 6\left[\left(\frac{1-\theta_I}{\Delta_T}\right) - \frac{\Delta}{\Delta_T}\right].$$

This system of coupled equations represents a complete mathematical solution to the problem for times $\tau < \tau_a$ and is adequate to determine the three unknown functions $\Delta(\tau, \varepsilon, \sigma)$, $\Delta_T(\tau, \varepsilon, \sigma)$, and $\theta_I(\tau, \varepsilon, \sigma)$, subject to the initial conditions that $\Delta(\tau = 0) = 0$, and $\Delta_T(\tau = 0) = 0$.

Once the thermal front, $\delta_T(t)$, reaches the wall surface, $x = -a$ (i.e. $\Delta_T = \Delta_{aT}$), the temperature at $x = -a$, T_a, is no longer constant; thus a different analysis must be furnished which is described as follows.

During the second stage of the analysis (that is, for $\tau > \tau_a$) the temperature distribution in the solidified crust is as described by equation (6), since the boundary conditions of the crust do not change. However, the temperature field in the wall is changed to satisfy the new boundary condition in the wall presented by equation (4), which is assumed as

$$\theta_w(x,t^*) = \theta_a - (\theta_a - \theta_I)\left(1 + \frac{x}{a}\right)^2. \quad (10)$$

Through use of equations (1), (2a), (3)–(6) and (10) the following new set of coupled equations is obtained:

$$\varepsilon\frac{d\Delta}{d\tau^*} = 2\left(\frac{1-\theta_I}{\Delta}\right) - 2\sigma\left(\frac{\theta_I - \theta_a}{\Delta_{aT}}\right) - 1,$$

$$\frac{d\theta_a}{d\tau^*} = \frac{1}{5}\left[12\left(\frac{\theta_I - \theta_a}{\Delta_{aT}^2}\right) - \frac{d\theta_I}{d\tau^*}\right],$$

and (11)

$$\frac{d\theta_I}{d\tau^*} = b_{10}\frac{d\Delta}{d\tau^*} + b_{20}\frac{d\theta_a}{d\tau^*} - b_{30},$$

where

$$b_{10} = b_1/b_0, \quad b_{20} = b_2/b_0, \quad b_{30} = b_3/b_0,$$

$$b_0 = \Delta\left[\frac{3}{2}\frac{\Delta}{\Delta_{aT}} + \left(\frac{\Delta}{\Delta_{aT}}\right)^2\right],$$

$$b_1 = 3\left(\frac{\Delta}{\Delta_{aT}}\right)\left\{(2\varepsilon+1) - \theta_I\left[1 + \sigma\left(\frac{\Delta}{\Delta_{aT}}\right)\right] + \sigma\frac{\Delta}{\Delta_{aT}}\theta_a\right\},$$

$$b_2 = \Delta\sigma\left(\frac{\Delta}{\Delta_{aT}}\right)^2, \quad (12)$$

and

$$b_3 = 6\left[\frac{1-\theta_I}{\Delta_{aT}} - \frac{\Delta}{\Delta_{aT}}\right].$$

The coupling of this set of equations is sufficient to determine the three unknown functions, in this case, $\Delta(\tau^*, \varepsilon, \sigma, \Delta_{aT})$, $\theta_I(\tau^*, \varepsilon, \sigma, \Delta_{aT})$, and $\theta_a(\tau^*, \varepsilon, \sigma, \Delta_{aT})$, subject to the initial condition $\Delta(\tau^* = 0) = \Delta(\tau_a)$ and $\theta_a(\tau^* = 0) = 0$. The initial value for the frozen crust, $\Delta(\tau^* = 0)$, and that for the interface temperature, $T_I(\tau^* = 0)$, are as determined during the first stage of the analysis. Numerical integration of the coupled equations was performed using the Gear method [13] (available from the Argonne National Laboratory Code Center).

4. DISCUSSION AND CONCLUSIONS

The (one-dimensional) behavior of the frozen crust that forms on an insulated, non-melting finite wall was studied using the refined integral heat balance method and found to depend on three parameters, namely; the wall–crust thermal ratio, σ, Stefan number for freezing, SN, and the dimensionless wall thickness, Δ_{aT}. As evidenced by Fig. 2(a), the wall thickness does not greatly influence the growth rate of the solidified crust at early times. The maximum thickness of the crust and its decay behavior, however, are strongly influenced by the wall thickness as long as the thermal front reaches the wall surface ($x = -a$) before the crust reaches its maximum thickness. Physically, the temperature at the opposite surface of the wall, T_a, begins to increase with time when the thermal layer, $\delta_T(t)$, reaches the wall surface. Eventually, T_a approaches the fusion temperature of the liquid, T_f, when the deposited crust completely disappears due to remelting. An evaluation of the total lifetime of the crust can simply be obtained from the energy balance $h(T_b - T_f)\,\tau_{\text{life}} = aCp_w\rho_w(T_f - T_0)$, or $\tau_{\text{life}} = \sigma\Delta_{aT}$, which is accurate when the cold wall is thin [11], [that is, when $\tau_{life} > a^2/(\alpha_w \pi)$ or when $\sigma > (\Delta_{at}/\pi)$].

Figures 2(b) and (c) show that the maximum crust thickness and the crust total lifetime are strongly dependent upon the wall–crust thermal ratio, σ, and the wall thickness, Δ_{aT}. An increase in σ or Δ_{aT} results in an increase in the maximum thickness of the crust and its total lifetime. Such behavior is expected since increasing the wall thickness or the wall–crust thermal ratio results in a more efficient wall heat sink, over a longer period of time. The Stefan number of freezing, SN, however, strongly influences Δ_{max} whereas it has a slight effect on τ_{life}. Increasing Stefan number increases the maximum crust thickness and reduces its total lifetime, due to faster growth and decay processes, as would be expected for materials with a lower heat of fusion.

The result of such a parametric study indicates that the wall thickness plays an important role in the transient growth and decay characteristics of the solidified crust; thus, previously obtained results for semi-infinite geometry are applicable only to problems where the growth and decay behavior of the crust occurs for times less than that for the thermal penetration through the solid wall (that is, for $\Delta_{aT} > \sigma\pi$). The solution technique presented here [10] can easily be extended to a wide variety of problems of engineering utility, where one wishes to obtain an estimation of the transient growth and decay behavior of a frozen crust that forms in a flowing liquid on a cold wall of finite extent. As discussed in [1, 12] such problems are of interest for nuclear reactor safety assessment, where an understanding of molten fuel freezing potential on core substructures is desirable for postulated reactor core overheating situations where fuel pin melting occurs.

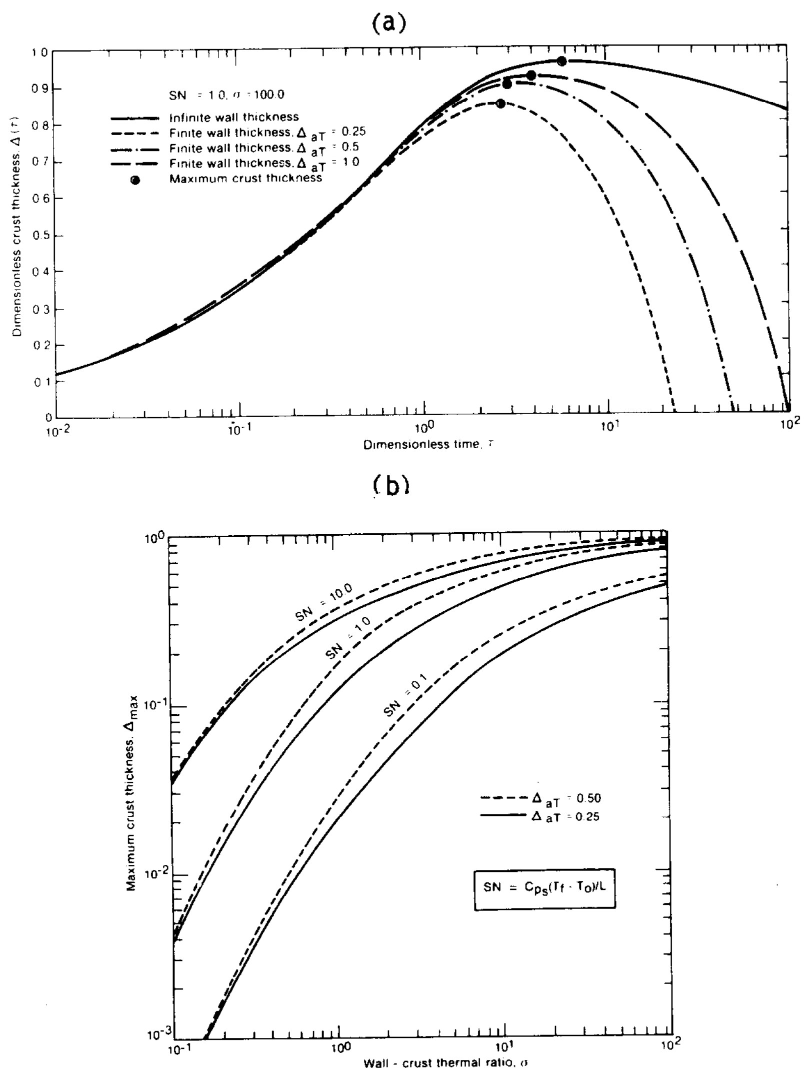
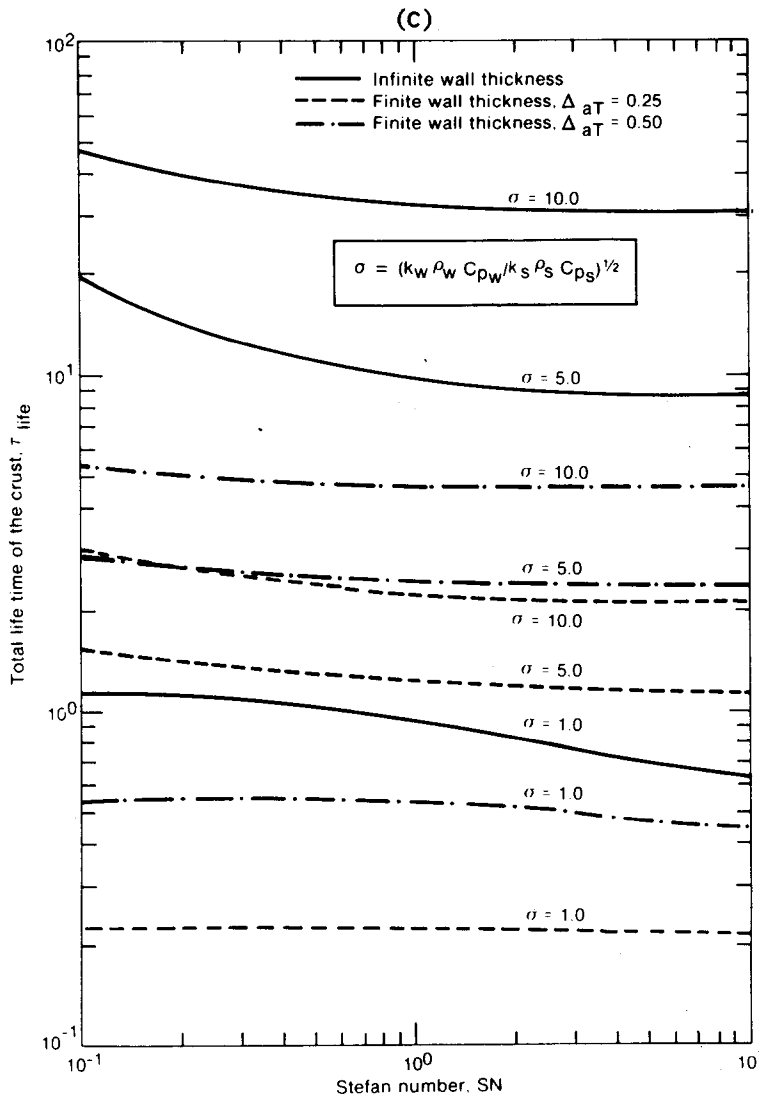


FIG. 2. The effect of (a) wall thickness, (b) Stefan number, and (c) wall–crust thermal ratio on the behavior of the solidified crust.



REFERENCES

1. M. Epstein, The growth and decay of a frozen layer in forced flow, *Int. J. Heat Mass Transfer* **19**, 1281–1288 (1976).
2. T. R. Goodman, The heat balance integral and its application to problems involving a change of phase, *J. Heat Transfer* **80**, 335–342 (1958).
3. J. M. Savino and R. Siegal, An analytical solution for solidification of a moving warm liquid onto an isothermal cold wall, *Int. J. Heat Mass Transfer* **42**, 803–809 (1969).
4. C. Lapadula and W. K. Mueller, Heat conduction with solidification and a convective boundary condition at the freezing front, *Int. J. Heat Mass Transfer* **9**, 702–704 (1967).
5. R. T. Beaubouef and A. J. Chapman, Freezing of fluids in forced flow, *Int. J. Heat Mass Transfer* **10**, 1581–1587 (1967).
6. R. D. Zerkle and J. E. Sunderland, The effect of solidification in a tube upon laminar-flow heat transfer and pressure drop, *J. Heat Transfer* **90**, 183–190 (1968).
7. M. S. El-Genk and A. W. Cronenberg, An assessment of fuel freezing and drainage phenomena in a reactor shield plug following a core disruptive accident, *J. Nucl. Engng Design* **47**, 195–225 (1978).
8. M. N. Ozisik and J. C. Mulligan, Transient freezing of liquids in forced flow inside circular tubes, *J. Heat Transfer* **91**, 385 (1978).
9. P. M. Heertjes and Ir. Ong Tjing Gie, Crystalization of water by unidirectional cooling. *Br. Chem. Engng* **5**, 412–419 (1960).
10. M. S. El-Genk and A. W. Cronenberg, Some improvements to Stefan-like problems, *Int. J. Heat Mass Transfer* **22**, 167–170 (1979).
11. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd ed. Clarendon Press, Oxford (1959).
12. M. S. El-Genk, Improvement to the solution of Stefan-like freezing and melting problems, with application to LMFBR safety analysis, Ph.D. Dissertation, University of New Mexico, Albuquerque, NM (August 1978).
13. C. W. Gear, The numerical integration of ordinary differential equations of various orders, Argonne National Laboratory Report, ANL-7126 (1966).